ROAD ENGINEERING DIVISION MEETING

24 January, 1952

Brigadier A. C. HUGHES, C.B.E., T.D., B.Sc., M.I.C.E.. Chairman of the
Division, in the Chair

The following Paper was presented for discussion and, on the motion of
the Chairman, the thanks of the Division were accorded to the Author.

Road Paper No. 36

" Some Theoretical Aspects of Road Traffic Research " *
by
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SYNOPSIS

Some of the mathematical and statistical aspects of the disposition and behaviour
of road traffic which are of importance in research are considered. It is shown that
vehicles can simultaneously be regarded as distributed at random along a road and in
time. Frequency distributions of speed for a given traffic stream are of two kinds, one
associated with successive vehicles passing a point and the other with successive
vehicles along a road at an instant. The corresponding average speeds generally
differ by 6 to 12 per cent.

A formula is given for the frequency with which vehicles would overtake one
another if there were no interference with overtaking.

In considering the capacity of road systems it should be remembered that increases
in the amount of traffic (the flow) generally produce corresponding decreases in speed.
Capacity can be defined as " the flow which produces the minimum acceptable average
journey speed."

Delay and capacity are discussed in relation to traffic signals, and it is shown that
the shortest practicable cycle does not necessarily result in the minimum average delay.
The randomness of traffic can cause very long delays at traffic signals.

When there are alternative routes which a traffic stream can follow, it may divide
itself between them. The consequences of two possible rules governing this division
are considered.

" Before-and-after " studies are discussed, and a statistical test for the genuineness
of a difference in, say, average journey time " before " and " after " is given.   A method
of determining the number of observations required is also given.

INTRODUCTION

ROAD engineers and research workers are tackling a large variety of
problems directly concerned with the movement of vehicles. Their
methods include " before-and-after " studies of speed for changes in road
lay-out, control or traffic composition, general surveys of traffic conditions,
and investigations into the effect of such factors as parking, road width,

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right turns, and lane markings on the behaviour of traffic. A number of theoretical problems occur in this work, and the object of the present Paper is to consider some of these problems. The Paper is concerned with some of the mathematical and statistical aspects of the disposition and behaviour of traffic on open roads and at intersections, including the questions of capacity, the use of alternative routes, and the planning and interpretation of "before-and-after" studies. The treatment is mainly statistical because the variability of vehicle behaviour often demands this approach, but statistical ideas have been introduced from first principles so far as possible. An attempt has been made to illustrate the application of some of the theory, although this is not always possible.

A glossary of terms and a list of the symbols used most frequently in the Paper are given in Appendix I. In general, when formulae are quoted, no units are mentioned, since the relations hold between the physical quantities concerned. When interpreting a formula it is, of course, necessary to use the same units of length and time throughout.

The Value of a Theoretical Approach

It is not always appreciated that in a severely practical subject such as traffic engineering there is need for theory. However, the history of science suggests that progress in any field of research can best be achieved by a judicious mixture of practical experience, experiment, and theory. Although a particular problem, such as that of assessing the saving in vehicle-hours due to widening a given road, can be solved by direct observation, a theoretical background is required if the observations are to be economical and useful and their interpretation valid. Moreover, the effect of road widening can be studied at a number of sites each with a different set of conditions, but it is a theoretical problem to generalize the results to cover conditions not actually observed. Theory may also be useful in suggesting the type of relation to be expected and the order of the result.

It must be emphasized that theory should not be divorced from experiment. A purely theoretical approach to a traffic problem usually has to be very much simplified or else it involves very difficult mathematical analysis. But simplifying assumptions can be tested by experiment, and empirical relations can be found by applying theoretical methods to data.

FLOW AND SPEED OF TRAFFIC

A necessary condition for many theoretical investigations into road traffic problems is that the way in which vehicles are distributed along the road should be specified at all times considered. That is to say, a theoretical model of traffic is required. Now Adams [1] has shown that the times at which vehicles pass a given point, in the absence of a nearby intersection or other interruption of the traffic stream and in conditions of moderate

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[1] The references are given on p. 354.

flow,* may be regarded as a random series.   He also remarked that traffic could be regarded as a random distribution in space along the road. Following Adams, a series of events in time is defined as " random " when :

(a) each event is completely independent of any other event ;
(b) equal intervals of time are equally likely to contain a given number of events.

A random distribution in space is defined by the corresponding conditions in terms of space instead of time.

It may not perhaps be clear that a theoretical model of traffic can be set up in which it is randomly distributed in both time and space, but this is the case.   Consider a random series formed by the times $\ldots t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots$ at which vehicles pass a point O on the road.   Suppose, first, that all the vehicles have the same speed v.   Then at any instant t, the vehicles will be at distances $\ldots v(t-t_{-2}), v(t-t_{-1}), v(t-t_0), v(t-t_1), \ldots$ along the road, measured from O.   Since the space-intervals between these points are merely the original time-intervals multiplied by v, they form a random distribution in space.

Consider now a traffic stream composed of a number of subsidiary streams, in each of which all the vehicles are travelling at the same speed and form a random series.   Since the combination of two random series or distributions in the above sense is itself a random series or distribution, the whole stream is random in both space and time.   Moreover, the speeds of successive vehicles in space or in time form a sequence of random variables, each being independent of its predecessor.

Distribution of Speed in Time

Suppose that there are subsidiary streams, with flows $q_1, q_2, \ldots q_C$ and speeds $v_1, v_2, \ldots v_C$.   Let the total flow be given by

$$Q = q_1 + q_2 + \ldots + q_C$$
$$= \sum_{i=1}^{C} q_i$$

and let $\qquad\qquad f_1 = q_1/Q \,;\, f_2 = q_2/Q \ldots ;\text{ and } f_C = q_C/Q$

Then the numbers $f_1, f_2, \ldots f_C$ are the frequencies in time of vehicles whose

speeds are $v_1, v_2, \ldots v_i$ and $\displaystyle\sum_{i=1}^{C} f_i = 1$

Distribution of Speed in Space

Consider the subsidiary stream with flow q_i and speed v_i.   The average time-interval between its vehicles is evidently $1/q_i$, and the distance

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* The term " flow " is used in this Paper in the sense of quantity of traffic (vehicles per hour) in preference to " volume."

travelled in this time is v_i/q_i.   It follows that the density of this stream in space, that is to say, the number of vehicles per unit length of road at any instant (the concentration), is given by

$$k_i = q_i/v_i, \; i = 1, 2, \ldots C \qquad \ldots \ldots \quad (1)$$

The quantities $k_1, k_2, \ldots k_C$ represent the concentrations of vehicles in each individual stream and the total concentration is given by

$$K = \sum_{i=1}^{C} k_i$$

Putting $f_i' = k_i/K$ gives the frequencies $f_1', f_2', \ldots f_C'$ of $v_1, v_2, \ldots v_C$ in space.

The following table illustrates these ideas for some actual data.   In practice, speeds vary continuously, but they have been grouped in 4-mile-per-hour groups.

TABLE 1.—DISTRIBUTIONS OF SPEED IN SPACE AND TIME.   WESTERN
   AVENUE, GREENFORD (MIDDLESEX).   DUAL CARRIAGEWAY.   EAST-
   BOUND TRAFFIC ONLY

Speed range m.p.h.	Flow v.p.h.	Percentage in time	Concentration v.p. mile	Percentage in space
* 2– 5	1	0·3	0·3	1·7
6– 9	4	0·8	0·5	3·3
10–13	0	0·0	0·0	0·0
14–17	7	1·5	0·4	2·9
18–21	20	4·5	1·0	6·9
22–25	44	9·9	1·9	12·7
26–29	80	17·9	2·9	19·7
30–33	82	18·2	2·6	17·7
34–37	79	17·5	2·2	14·9
38–41	49	10·9	1·2	8·3
42–45	36	7·9	0·8	5·4
46–49	26	5·7	0·6	3·8
50–53	9	1·9	0·2	1·1
54–57	10	2·2	0·2	1·2
58–61	3	0·8	0·1	0·4
Total	450	100·0	14·9	100·0

*·Strictly $1\frac{1}{2}$–$5\frac{1}{2}$, $5\frac{1}{2}$–$9\frac{1}{2}$, etc.

The two distributions are illustrated in *Figs 1*.   It will be seen that there is a distinct difference between them ; the bulk of the time-distribution is further to the right than that of the space-distribution, meaning that speeds are greater on the whole for the time-distribution.   There will always be such a difference if there is any variation in speed.

Figs 1
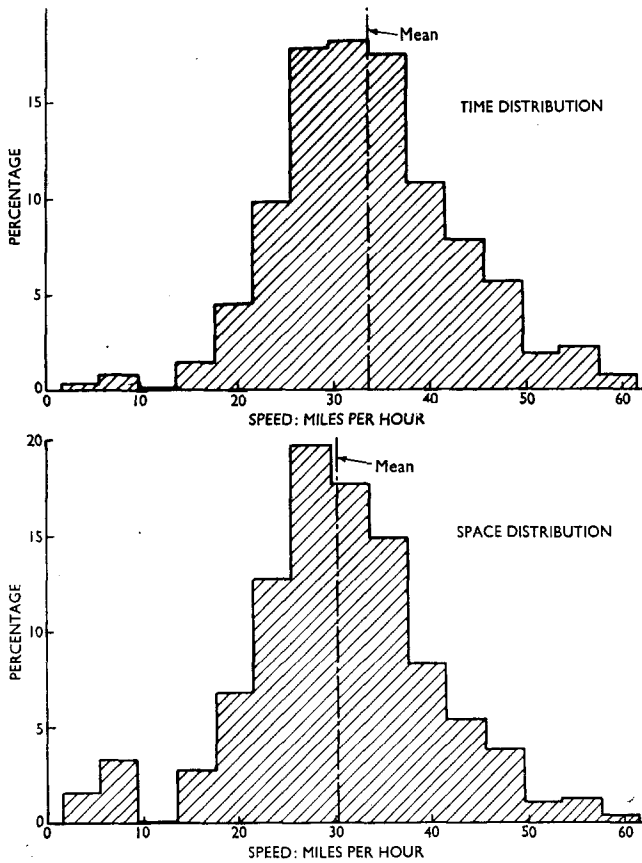
TIME AND SPACE DISTRIBUTIONS OF SPEED. WESTERN AVENUE, GREENFORD (DUAL CARRIAGEWAY)

Mean Speeds

With each of two distributions of speed there is associated a mean value. They are given by the following formulae :

$$\text{Time-mean speed}: \quad \bar{v}_t = \sum_{i=1}^{C} q_i v_i / Q = \sum_{i=1}^{C} f_i v_i \quad \cdots \cdots \cdots \quad (2)$$

$$\text{Space-mean speed}: \quad \bar{v}_s = \sum_{i=1}^{C} k_i v_i / K = \sum_{i=1}^{C} f_i' v_i' \quad \cdots \cdots \cdots \quad (3)$$

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But from equation (1)

$$k_i v_i = q_i \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and therefore
$$\bar{v}_s = \sum_{i=1}^{c} q_i / K = Q/K$$

Hence $Q = K\bar{v}_s$ $. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$

It may be noted that there is no equivalent relation involving the time-mean speed.

It is helpful to consider how these alternative mean speeds might arise in practice. Speeds at a point on the road can be measured by any of the following methods :

(1) Timing successive vehicles, or a representative selection (for example, every 5th, or those whose registration numbers end in 2), over a short measured distance of length l, and calculating $v = l/t$ for each, where t is the time taken. The mean is $\bar{v} = \Sigma v/n$ where n denotes the number of vehicles.

(2) Use of the Radar Speedmeter, which gives a direct reading of speed for each vehicle. The mean is $\bar{v} = \Sigma v/n$.

(3) Timing vehicles over a short distance as for (1), but calculating first the average time taken, $\bar{t}$, and then $\bar{v}' = l/\bar{t}$ as the mean.

(4) Taking two aerial photographs of a uniform road at a short interval τ, measuring the distance covered by each vehicle, x, and computing $v = x/\tau$. The mean is $\bar{v}' = \Sigma v/n$.

Of these methods, (1) and (2) give the time-mean speed, and (3) and (4) give the space-mean speed.

It is, of course, possible to calculate either the time- or the space-mean speed from a set of readings of speed, if it is known how they were obtained and therefore which type of distribution they form. Thus, if $v_1, v_2, \ldots v_n$ are obtained from a Radar Speedmeter they form a time-distribution, and the time-mean is $\Sigma v/n$, whilst the space-mean is given by

$$\bar{v}_s = \frac{n}{\Sigma(1/v)}$$

This is, of course, the harmonic mean of the v's. On the other hand, if they are obtained from aerial photographs or by a similar technique they form a space-distribution, $\Sigma v/n$ is the space-mean, and the time-mean is

$$\bar{v}_t = \frac{\Sigma v^2}{\Sigma v}$$

It is shown in Appendix II that the general relation between time- and space-means is

$$\bar{v}_t = \bar{v}_s + \frac{\sigma_s^2}{\bar{v}_s} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where σ_s is the standard deviation * of the space-distribution, defined by

$$\sigma_s^2 = \sum_{i=1}^{C} k_i(v_i - \bar{v}_s)^2/K \quad \ldots \ldots \quad (7)$$

and measured in the same units as the speed. If c_s is the coefficient of variation of the space-distribution of speed, defined as the ratio of the standard deviation to the mean (that is to say, $c_s = \sigma_s/\bar{v}_s$)

then
$$\bar{v}_t = \bar{v}_s(1 + c_s^2) \quad \ldots \ldots \ldots \ldots \quad (8)$$

As an illustration of this result, consider the example of Table 1, where the following values are found :

	Mean speed: m.p.h.	Standard deviation: m.p.h.	Coefficient of variation
Time :	33·5	9·1	0·27
Space :	30·1	10·1	0·34

It can be verified that equation (8) holds in this case. The equation shows that in every case the time-mean speed is greater than the space-mean speed, unless there is no variation in speed, when $c_s = 0$. The coefficient of variation is usually between 0·25 and 0·35 so that the time-mean is 6 to 12 per cent greater than the space-mean.

In a particular investigation, it may not be very important which of the two means is used, although it will be shown below that some cases call for one and some for the other. But it is most important that the same mean should be used throughout any investigation, so that all comparisons are fair. There is a danger that a comparison of mean speeds measured some years apart or by different investigators will be invalid because they are not of the same kind.

Speed versus Journey Time

The choice between time- and space-mean speeds is really one between speed and journey time. Fundamentally it is the characteristics of success-ive journeys on a given route that are required, and these can be analysed in terms of speed, giving the time-mean speed, or in terms of time, giving the mean journey time, which by division into the length of the route gives the space-mean speed.

Now time is a directly measurable physical quantity. On any route

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* In general, if a frequency distribution consists of quantities $x_1, x_2, \ldots x_C$ with frequencies $f_1, f_2, \ldots f_C$, and the mean is $\bar{x} = \sum_{i=1}^{C} f_i x_i$, the standard deviation σ is a measure of dispersion, defined by the equation $\sigma^2 = \sum_{i=1}^{C} f_i (x_i - \bar{x})^2$.

or section of road, the length is a fixed quantity, while the duration of the journey varies. Times on successive sections are additive whereas speeds are not. The delay caused by two equal intersections is twice the delay caused by one, provided that they are sufficiently far apart. In planning a journey, one wishes to know how long it will take rather than what the average speed will be. Also the cost of slow movement is measured in terms of time. There is therefore much to be said for using average times in preference to average speeds when assessing the effect of a change.

Figs 2
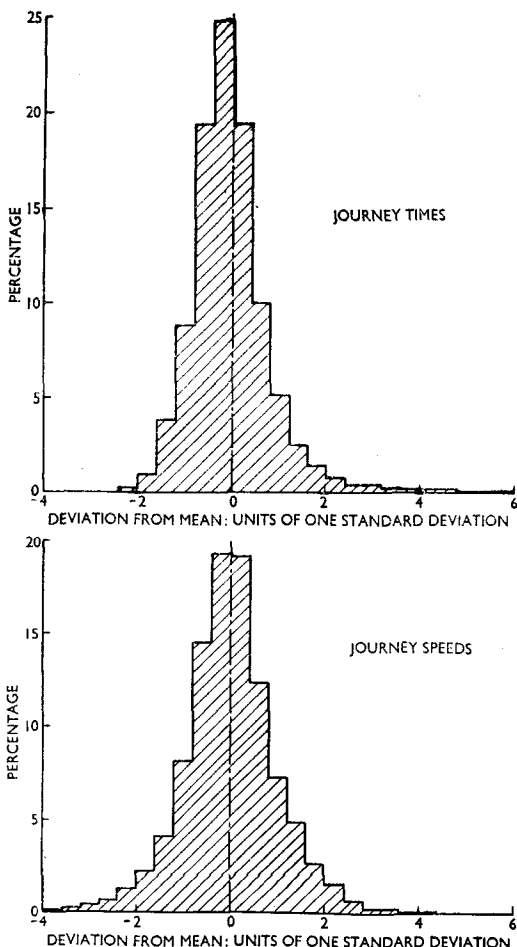
FREQUENCY DISTRIBUTIONS OF JOURNEY TIME AND SPEED IN CENTRAL LONDON

On the other hand speed measurements are frequently more consistent. It has been found that journey times often have very skew distributions with a long " tail " consisting of very slow journeys, whereas the corresponding distributions of speed tend to be more symmetrical. A typical example is shown in *Figs 2*, where the frequency distributions of journey time and journey speed for roads in Central London are plotted. It was also found from these data that the coefficient of variation was 20 to 25 per cent less in the case of speeds. This is an important result, since it means that only about 60 per cent of the number of runs are needed to give the same proportional accuracy in the mean speed as in the mean journey time.

The results quoted above were obtained on main roads in London, with a high concentration of controlled intersections (about 5 per mile). The more intersections there are per mile the greater is the tendency to skewness in the journey-time distribution. Where intersections are less dense, the journey-time distribution may be reasonably symmetrical, and so little is to be gained by converting to speeds.

FREQUENCY OF OVERTAKING

Assuming that there is no interference with overtaking, the frequency with which vehicles overtake one another can be derived from the speed distribution. Suppose that $v_1 < v_2 < \ldots < v_C$, and consider a vehicle with speed v_i. The speed relative to it of a subsidiary stream whose speed is v_j, where $j > i$, is $v_j - v_i$, and their concentration is k_j. Hence the average distance between them is $1/k_j$, and therefore they pass the vehicle with speed v_i at intervals of $1/(v_j - v_i)k_j$. Hence the frequency of overtaking experienced by this vehicle is

$$\sum_{j=i+1}^{C} k_j(v_j - v_i),$$

the summation extending over all speeds faster than v_i. Now there are k_i of these vehicles per unit length of road, each being overtaken at the average rate given above. Hence the total number of overtakings in which the slower vehicle has speed v_i is $k_i \sum_{j=i+1}^{C} k_j(v_j - v_i)$ per unit length per unit time. Summing over $i = 1, 2, \ldots C$ gives the total number of overtakings as $\sum_{i=1}^{C-1} \sum_{j=i+1}^{C} k_i k_j(v_j - v_i)$ per unit length per unit time. In terms of the frequencies in space (f_i') and time (f_i) respectively, this becomes :

$$\text{number of overtakings} = K^2 \sum_{i=1}^{C-1} \sum_{j=i+1}^{C} f_i' f_j'(v_j - v_i) \text{ per unit length per unit time}$$

$$\ldots \quad (9)$$

$$= Q^2 \sum_{i=1}^{C-1} \sum_{j=i+1}^{C} f_i f_j \left(\frac{1}{v_i} - \frac{1}{v_j} \right) \begin{matrix} \text{per unit length} \\ \text{per unit time} \end{matrix} \quad . \quad . \ (10)$$

If there is interference with overtaking, expression (10) can be taken as the number of desired overtakings per unit length per unit time, provided that the distribution of speeds is determined when the flow is low, so that it represents desired speeds. Thus for a given distribution of speeds the number of desired overtakings increases as the square of the flow.

In the particular case where the space-distribution of desired speed is approximately " normal " *, with standard error σ_s, by replacing the discontinuous distribution of speeds by a continuous one the density of overtaking can be expressed as :

$$\text{number of overtakings} = \frac{Q^2 \sigma_s}{\bar{v}_s^2 \sqrt{\pi}} = 0.56 \frac{Q^2 \sigma_s}{\bar{v}_s^2} \text{ per unit length} \atop \text{per unit time} \quad . \quad (11)$$

For instance, in the example quoted in Table 1, assuming that the space-distribution is approximately normal, there would be a density of desired overtakings of 1,090 per mile per hour. The actual number, of course, would generally be smaller owing to interference, and the ratio of actual to desired overtakings could be regarded as an index of traffic congestion.

CAPACITY OF ROAD SYSTEMS

The subject of the capacity of road systems has been discussed for many years. Numerous estimates of the capacity of a traffic lane have been based on the assumption that each vehicle follows the one ahead at exactly the same speed and at the minimum safe headway or the minimum headway acceptable to the average driver. However, these conditions are extremely artificial and take little account of what drivers actually do. They may apply to some extent at intersections but on a section of road free from controlled intersections entirely different considerations apply. An alternative approach is needed ; references to a suitable method have already been made by Glanville [2,3] and Smeed,[4] but this method will be discussed more fully here.

Owing to the randomness of the traffic and the variability of its speeds, in moderate traffic there is a continual shuffling process and overtaking is frequent. If the flow is greater, there is more interference with overtaking, and the average speed is less. It is more appropriate to regard the speed as a function of the flow than vice versa.

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* The " normal" frequency distribution is symmetrical about a single peak, and if $\bar{x}$ is the mean and σ the standard deviation the frequency in the range x to $x + dx$ is
$$\frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}(x-\bar{x})^2/\sigma^2} dx.$$

This point of view may be unfamiliar to road engineers, and perhaps requires some elaboration.   One difficulty is that as congestion increases a time is reached when traffic stops completely, although only temporarily. At such a time the speed is zero and so is the flow, a state of affairs which does not conform with the idea of speed decreasing as flow increases. However, average conditions are more important than chance fluctuations in the long run.   It is helpful to consider an actual experiment to determine the relation between speed and flow on a given section of road in one direction.   Any delays caused by conditions in this section ought to be included, although some of these delays may occur outside it.   The experiment can only be made satisfactorily if there is a sufficiently free approach, which will not become congested before the section being studied does, and which can accommodate all vehicles held up by congestion in the section. Also there must not be a bottleneck beyond the section which can react on it.   These are extremely practical problems when heavily intersected roads are being studied and make the isolation of road sections very difficult.   However, suppose that it can be done, and that either the inflow to the section can be controlled or that it varies with time in a suitable way. Then by taking a timing point on the approach road well outside the section, so that all vehicles waiting to enter it are included, and another timing point at the end of the section, values of average speed (or journey time) and inflow can be obtained for a series of periods.   If the flow increases the average speed will generally fall (sometimes imperceptibly at first—see below).   Eventually a stage will be reached at which the slightest increase in flow will cause a queue * to accumulate at some point in the section.   This may be called the saturation level of flow.   It could be taken as the capacity of the road, but as it corresponds to a position of instability, and may produce very low speeds indeed, this is scarcely appropriate.

In practice, of course, it may be impossible to isolate a road section in the way described.   If so, it is not possible to define its capacity since its behaviour is governed by that of neighbouring sections.   In such a case the capacity of a longer section should be considered.

Capacity of an Open Street
An example of the results actually obtained is shown in *Fig. 3* where values of running speed (speed while in motion) are plotted against the flow on the whole street for groups of main streets of varying width in Central London.   These results have been given previously by Glanville.[3] It will be seen that, in general, as the flow increases the speed is not appreciably influenced at first, but beyond a certain point the speed decreases steadily with increasing flow.   In the case of the narrow streets the speed begins to decrease at once.   It has been found possible to approximate to these results by the following empirical formula :

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* A queue is here understood to be a chain of vehicles at minimum headway whether stationary or moving.

$$v_r = 31 - \frac{q + 430}{3\,(w - 6)} \text{ or 24 m.p.h., whichever is less,}$$

where v_r denotes space-mean running speed in miles per hour.
            q ,,   total flow in vehicles per hour
            w ,,   road width in feet,

provided that the speed given by the formula is not less than 10 miles per hour. However, it must be regarded as a convenient accident that the speed/flow relation is approximately linear within certain wide ranges. Normann [5] gives a single straight-line relation between speed and flow on four-lane open divided roads in the U.S.A. throughout the range from zero

Fig. 3
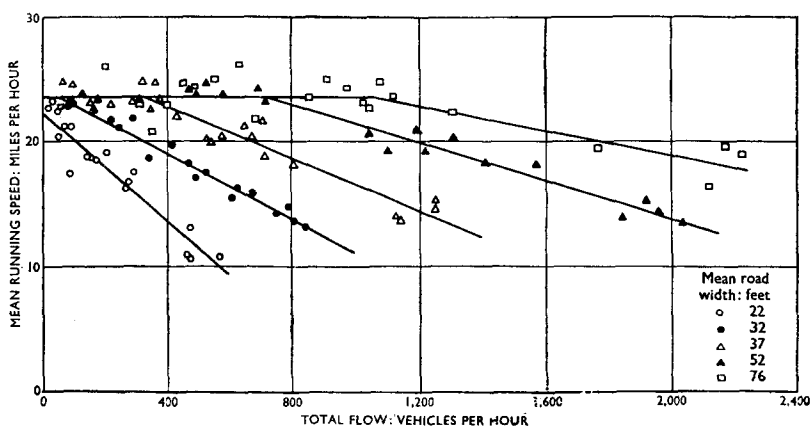
RELATION BETWEEN MEAN RUNNING SPEEDS AND TOTAL FLOW IN CENTRAL
LONDON

to 4,000 vehicles per hour, but since he gives no points it is not possible to judge how well this relation fits the facts.

Given the relation between speed and flow, capacity can be defined as " the flow which produces the minimum acceptable journey speed." The average capacity so defined can be calculated for these Central London streets from the above formula. For instance, if the minimum acceptable speed is 15 miles per hour, the average capacity of a 30-foot street is about 700 vehicles per hour. In other words, if 700 vehicles require to use such a street, conditions will allow them to travel at an average speed of 15 miles per hour. The corresponding average capacities for 40- and 50-foot roads are 1,200 and 1,700 vehicles per hour respectively.

Theoretical Approach to Capacity of an Open Road

By the use of the ideas mentioned earlier, and some assumptions about the time required to overtake a vehicle, it should be possible to deduce a

theoretical speed/flow relation for an open road. Some work on these lines has been done by Kinzer,[6] but he restricted himself to the question of the distance which a vehicle can expect to travel without interference. This is clearly related to the average speed, but more work is required before the speed itself can be found. It appears that the problem could be solved in a particular case by a somewhat tedious computation, but the derivation of a general expression appears to be difficult.

SIGNAL-CONTROLLED INTERSECTIONS

In the case of an intersection, theoretical treatment is easier, and Clayton[7] has given several formulae for cycle time, capacity, and delay at traffic signals. By following his calculations with slightly more general values, the results given below for a 2-phase signal can easily be obtained. It is assumed that the intersection and its traffic are " symmetrical " ; that is to say, that the flow from, say, the north equals that from the south and the corresponding approach roads are of equal width, and similarly for the east-west direction. It is also assumed that the flow is uniform, that is to say, the intervals between the successive vehicles are all the same for the period studied. Let the following symbols be defined :

	Phase 1	Phase 2	Total
Flow	q_1	q_2	Q
Saturation flow .	p_1	p_2	
" Lost " time .	a_1	a_2	A
" Green " time .	g_1	g_2	$c =$ cycle time
" Red " time .	$r_1 = g_2$	$r_2 = g_1$	

These symbols can be interpreted by reference to *Fig. 4*. On each phase, vehicles arrive with inflow q, the appropriate subscript being added, and hence at intervals of $1/q$. Some are stopped during the red period, of length r, including the red/amber. At the green signal, the first vehicle leaves, but it is delayed by an additional amount a due to its acceleration. After this vehicle, however, successive vehicles follow at equal intervals of $1/p$ until no more are waiting. This is the period when vehicles are following one another at minimum achievable headway, and the flow p is called the " saturation flow." During the remainder of the " green " period vehicles arrive and leave at intervals of $1/q$, and experience no delay.

With this notation q and p take the place of Clayton's D, the density of traffic (flow per lane), and S, the saturation density. In fact, if there are n lanes, $q = nD$ and $p = nS$. It is more convenient to work in terms of flow rather than density, because traffic does not always divide up into a fixed number of lanes, even at a controlled intersection. The formula given by Clayton[7] for the minimum cycle which can accommodate the traffic without the formation of a steadily increasing queue on either phase, generally known as Adams' Formula, becomes

Fig. 4
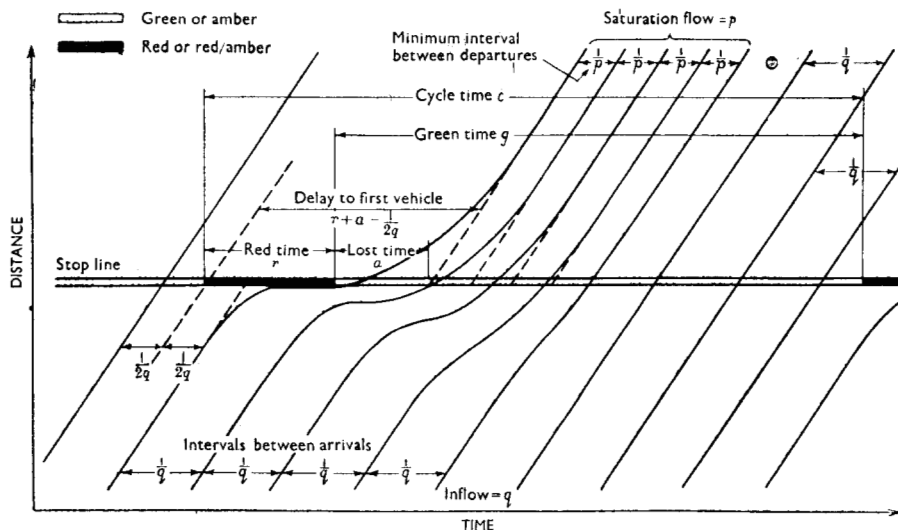
DISTANCE/TIME DIAGRAM FOR SIGNAL-CONTROLLED INTERSECTION

$$c = \frac{A}{1 - \dfrac{q_1}{p_1} - \dfrac{q_2}{p_2}} \quad \cdots \cdots \cdots \quad (12)$$

A formula for the average delay to traffic on phase 1 is

$$t_1 = \frac{\left(r_1 + a_1 - \dfrac{1}{2p_1}\right)^2}{2c\left(1 - \dfrac{q_1}{p_1}\right)} \quad \cdots \cdots \quad (13)$$

This is almost identical with that given by Clayton; for the sake of completeness the derivations of both of the above expressions are given in Appendix III.

In the expression for average delay quoted above, the term $1/2p_1$ is generally small compared with $r_1 + a_1$ and can be neglected. Hence the average delays on each phase on a 2-phase signal can be approximately represented by

$$\left.\begin{aligned} t_1 &= \frac{(r_1 + a_1)^2}{2c\left(1 - \dfrac{q_1}{p_1}\right)} \\[2em] t_2 &= \frac{(r_2 + a_2)^2}{2c\left(1 - \dfrac{q_2}{p_2}\right)} \end{aligned}\right\} \quad \cdots \cdots \quad (14)$$

The quantities $r_1 + a_1$, $r_2 + a_2$ may be called the " effective red times " for the two phases.

The average delay for the intersection as a whole is given by

$$T = (q_1 t_1 + q_2 t_2)/Q \quad \ldots \ldots \quad (15)$$

the average delay on each phase being weighted by the appropriate flow

Hence
$$T = \frac{1}{2Qc} \left\{ \frac{q_1(r_1 + a_1)^2}{1 - \dfrac{q_1}{p_1}} + \frac{q_2(r_2 + a_2)^2}{1 - \dfrac{q_2}{p_2}} \right\} \quad . \quad (16)$$

Optimum Phase Times for a Two-Phase Fixed-Time Signal

A reasonable criterion for the optimum settings of a traffic signal is that the average delay to all vehicles should be the minimum. In most practical examples the optimum cycle in this sense is also the minimum cycle. However, it is shown in Appendix IV that this is not necessarily the case. The conditions in which it is so are given, and in the other cases the cycle which gives the minimum average delay is specified. An example of the latter case, in which the flow is assumed to be uniform, is given below :

	Phase 1	Phase 2	Total
Lost time (a), secs. . . .	10	10	20
Flow (q), v.p.h.	1,000	200	1,200
Saturation flow (p), v.p.h. .	5,000	2,000	

Comparison of Minimum and Optimum Cycles

		Time : seconds	
Phase 1	Phase 2	Minimum cycle	Optimum cycle
Green	Red . .	10	12
Amber	R/Amber.	3	3
Red	Green .	13	33
R/Amber	Amber .	3	3
Cycle time		29	51
Mean delay : phase 1 .		11	8
Mean delay : phase 2 .		13	23
Overall mean delay. .		11·6	10·3

It will be seen that the average delay can be reduced by 11 per cent if the cycle is increased by 75 per cent. This sort of situation may arise when one phase refers to a minor road. It is not known whether a similar situation can arise in the case of random traffic.

Vehicle-Actuated Signals

Clayton [7] has given a formula for the average delay to a light flow of vehicles on, say, a minor road at vehicle-actuated signals, if the traffic is

random but there is no " maximum green " period on the major road. Garwood [8] has given the average delay to the same vehicles in the more general case where traffic is random and there is also a " maximum green " period on the main road. In the general case the problem of calculating average delay becomes very complicated mathematically and so far no solution has been worked out.

It is possible to give a very approximate solution assuming uniform flow and also an " ideal " vehicle-actuated signal. It is assumed that this signal adjusts itself to the minimum cycle as the flow varies. Then it is shown in Appendix V that in the case of a symmetrical intersection, where $q_1 = q_2 = q$, $p_1 = p_2 = p$ and $a_1 = a_2 = a$, the average delay per phase is given by

$$t_v = \frac{a\left(1 - \frac{q}{p}\right)}{1 - \frac{2q}{p}} \qquad \ldots \ldots \ldots (17)$$

This can be compared with the corresponding formula for a fixed-time signal obtained from (14):

$$t_f = \frac{(r + a)^2}{2c\left(1 - \frac{q}{p}\right)} \qquad \ldots \ldots \ldots (18)$$

These two formulae for delay are plotted against flow for a particular example in *Fig. 5*. It will be seen that the vehicle-actuated signal is very much better than the fixed-time variety until saturation is nearly reached.

The assumption made above about the behaviour of the vehicle-actuated signal is most applicable to a heavily trafficked signal controlled by a traffic integrator which adjusts the cycle in relation to the number of vehicles entering the intersection in accordance with Adams' Formula, although this type is usually used only as the centre of a progressive system, as at Baker Street/Marylebone Road in London.

The Effect of Randomness of Traffic on Intersection Delay

The two formulae for delay quoted above, (17) and (18), depend on the assumption of uniformity of flow. It is a fact of experience, however, that average delays at signals are longer than would be expected on the basis of these formulae at high levels of flow. This is easily explained, since random fluctuations in the number of vehicles arriving per cycle mean that on some occasions a surplus is left over to the next cycle, and delays are considerably increased. Although the average flow is less than the maximum and fluctuations are no more than random, it is quite possible for vehicles to be held for as many as four cycles.

It has been mentioned that a full mathematical treatment of random flow is difficult although, as Adams [1] has shown, it is possible to solve a large

Fig. 5

Cycle = 60 seconds = 30 seconds per phase
(maximum in vehicle-actuated case)
Lost time = 12 seconds
Saturation flow per 10-foot width = 1,200 v.p.h.
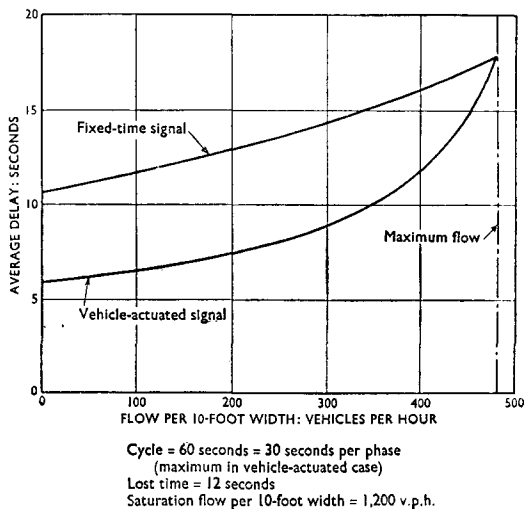
THEORETICAL VARIATION OF DELAY WITH FLOW, GIVEN UNIFORM FLOW, AT A
TRAFFIC SIGNAL

variety of specific numerical problems. Greenshields[9] gives a step-by-step process for building up a solution, but admits that it is not practical. As an interim method, a partially theoretical method using random numbers could be very useful. As an indication of the results which could be obtained by this method, an investigation has been made in which random arrival times for 2,000 vehicles were obtained from random numbers. A fixed cycle was assumed, consisting of 100 seconds equally divided between green and red and with a lost time of 20 seconds. In order to simplify the computation the average delay was calculated for six values of $1/p$ (the interval between departing vehicles when the flow is at the saturation level). The results are shown in *Fig. 6* compared with the expected values on an assumption of uniform flow obtained from formula (14). The agreement is good up to the point which corresponds to 50 per cent of saturation, and even at 75 per cent of saturation the uniform-flow value is only about 20 per cent too low. But as the interval between departing vehicles, and hence the degree of saturation, increases further, the average delay increases more and more rapidly in the random case.

This example shows how delay is increased by increases in $1/p$, that is to say, by reductions in p, the saturation flow. Curves of a similar type would be expected if q, the flow, were to be increased while p remained fixed. Some examples of the average effect of changes in flow on London streets are shown in *Fig. 7*. These curves were obtained by combining the data from a number of controlled intersections.

Fig. 6

Corrigendum :
for " 900 v.p.h."
read " 360 v.p.h."
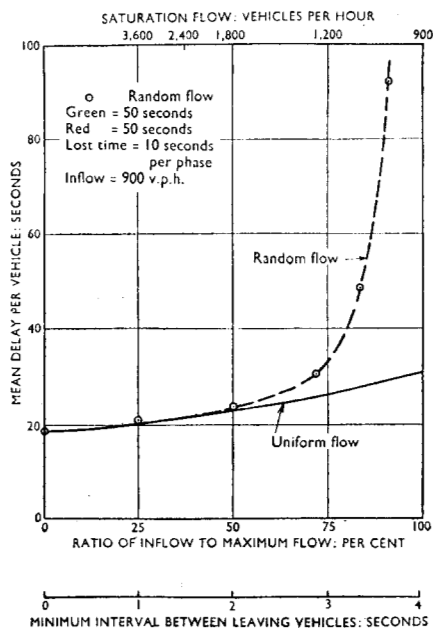
MEAN DELAY AT A FIXED-TIME TRAFFIC SIGNAL.   RANDOM AND UNIFORM FLOW

Fig. 7
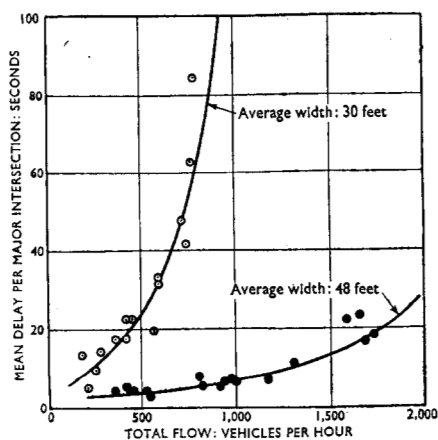
DELAY PER MAJOR INTERSECTION ON TWO GROUPS OF STREETS IN CENTRAL
LONDON

FORMATION OF QUEUES

A question related to that of delay is that of the rate at which a queue of vehicles will form when there is some obstruction, such as an intersection, a bottleneck, or an accident. Suppose that vehicles are arriving with flow q and space-mean speed v, and that once they have joined the queue their average flow is q_0 and their average speed v_0. Evidently if $q < q_0$, no queue forms, excluding temporary queues which oscillate in length. If $q > q_0$, a steadily increasing queue forms; let its length be x at time t, and suppose that $x = 0$ when $t = 0$. Consider the stretch of road of length x leading to the intersection. At $t = 0$ it contained kx vehicles, where $k = q/v$. At time t it contains $k_0 x$ vehicles, where $k_0 = q_0/v_0$, since it is filled with vehicles in the queue.

Now the difference between these two numbers equals the difference between the number entering the stretch and the number leaving it, $(q - q_0)t$. Hence $(k_0 - k)x = (q - q_0)t$.

If u is the rate at which the queue lengthens, then:

$$u = \frac{dx}{dt} = \frac{q - q_0}{k_0 - k}$$

that is to say:

$$u = \frac{q - q_0}{\dfrac{q_0}{v_0} - \dfrac{q}{v}} \qquad \ldots \ldots \ldots \quad (19)$$

In the case of a signal-controlled intersection with effective red time $r + a$, cycle time c, and saturation flow p, it can be shown, by considering the distance travelled by a vehicle in one cycle, that the average speed in the queue is:

$$v_0 = \frac{1}{\dfrac{1}{v} + \dfrac{k_1(r + a)}{p(c - r - a)}} \qquad \ldots \ldots \quad (20)$$

where k_1 denotes the concentration of stationary vehicles. The average flow of the queue is $q_0 = p(- r - a)/c$.

If there is a complete blockage, $q_0 = 0$ and $k_0 = k_1$ (the concentration of a stationary queue). Hence the queue lengthens at the rate:

$$u = \frac{q}{k_1 - \dfrac{q}{v}} \qquad \ldots \ldots \ldots \quad (21)$$

As an example of the use of this formula, consider a single lane which is completely blocked, and in which vehicles form up with a headway of 18 feet. If the vehicles arrive with a speed of 15 miles per hour and a flow of 500 vehicles per hour, the queue increases at 1·9 miles per hour.

DISTRIBUTION OF TRAFFIC OVER ALTERNATIVE ROUTES

When the effect of some future improvement of a road system is to be judged, some estimate must be made of the distribution of traffic on the various roads affected, including not only new roads but all existing roads from which traffic may be diverted. This is usually done by making some rather arbitrary assumption about speeds on the new roads, and, given the results of an Origin and Destination survey, by assuming that every vehicle will travel by the quickest route. However, it has been seen that speed is a function of flow, so that redistribution of traffic upsets the pattern of speeds. The problem is to discover how traffic may be expected to distribute itself over alternative routes, and whether the distribution adopted is the most efficient one. Although there has not been a sufficiently detailed investigation of a road network to allow this to be done in practice, it seems worth while to consider the theoretical aspects of this problem.

Consider the case of a given flow of traffic Q, which has the choice of D alternative routes from a given origin to a given destination, numbered $1, 2, \ldots D$. Suppose that the flow Q is divided amongst them so that a quantity q_i follows route i. Then:

$$q_i \geqslant 0, \ i = 1, 2, \ldots D \text{ and } \sum_{i=1}^{D} q_i = Q$$

The problem is to determine the q_i subject to these conditions.

Suppose that the speed/flow relation is linear on each route in the range of flows considered, and that the flow on route i is q_i' before the addition of q_i. Then, if v_i is the speed on route i, $v_i = a_i - a_i'(q_i' + q_i)$ where a_i and a_i' are constants for route i, depending on street widths and intersections, etc. If l_i is the length of the route, the journey time is:

$$t_i = \frac{l_i}{a_i - a_i'q_i' - a_i'q_i}$$

This may be written in the simpler form:

$$t_i = \frac{b_i}{1 - \dfrac{q_i}{p_i}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where b_i and p_i are new constants depending on l_i, a_i, a_i' and q_i'. It is interesting to note the similarity of this expression to that for average delay at fixed-time signals (14).

Consider two alternative criteria based on these journey times which can be used to determine the distribution on the routes, as follows:

(1) The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

(2) The average journey time is a minimum.

The first criterion is quite a likely one in practice, since it might be assumed that traffic will tend to settle down into an equilibrium situation in which no driver can reduce his journey time by choosing a new route. On the other hand, the second criterion is the most efficient in the sense that it minimizes the vehicle-hours spent on the journey. In practice, of course, drivers will be influenced by other factors, such as the state of the roads, and the comfort or discomfort of driving in general. However, it is clearly difficult to allow for these psychological factors.

Consider each criterion separately.

(1) *Equal Times*

The problem can be stated as follows: Given Q, b_1, b_2, ... b_D, p_1, p_2, ... p_D and the relation (22), which routes must be used so that the journey time is the same on each, equal to t, say, and t is less than the journey time which would be experienced by a single vehicle on any route not used? Also, what is the value of t?

Suppose that $b_1 < b_2 < \ldots < b_D$. Any one of these quantities b_i is the journey time on route i when the additional flow $q_i = 0$. Suppose that $b_j < t > b_{j+1}$ for some route number j. Then clearly only the first j routes can be in use, for equation (22) shows that the journey time on any route i cannot be less than b_i. On the other hand, all of the first j routes must be in use; otherwise t would be greater than b_i for a route i which was not used. It follows that:

$$q_i = p_i\left(1 - \frac{b_i}{t}\right) \quad \text{(where } i = 1, 2, \ldots j\text{).} \quad . \quad . \quad (23)$$

these equations being obtained by re-writing (22). Summing over the first j routes gives:

$$Q = \sum_{i=1}^{j} p_i - \frac{1}{t}\sum_{i=1}^{j} p_i b_i \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This equation gives the value of Q for which t is the appropriate journey time. If Q is calculated for each value of t, it is possible to find the solution to the problem by picking out the value of t which corresponds to the given Q.

(2) *Minimum Average Time*

In this case the problem is as follows :—Given Q, b_1, b_2, ... b_D, p_1, p_2, ... p_D as before, find q_1, q_2 ... q_D satisfying the conditions such that the average journey time T is a minimum.

23

Now
$$T = \sum_{i=1}^{D} q_i t_i / Q \quad \ldots \ldots \ldots \quad (25$$

Note that if z_i is the average number of additional vehicles on route i at an instant, then $z_i = q_i t_i$. Let $Z = \sum_{i=1}^{D} z_i$ be the total number of additional vehicles on the routes.

Then
$$T = Z/Q \quad \ldots \ldots \ldots \quad (26)$$

Now the criterion considered is that T should be a minimum. Since Q is constant, this is equivalent to Z—the total number of additional vehicles *en route* being a minimum.

Suppose that the solution has an additional flow q_i on route i, $i = 1, 2, \ldots D$. Then if i and j are any two routes in use, the value of Z must not be altered by the transfer of an infinitesimal increment of flow δq from route i to route j. This means that :

$$\frac{d(q_i t_i)}{dq_i} = \frac{d(q_j t_j)}{dq_j} \quad \ldots \ldots \ldots \quad (27)$$

Hence
$$\frac{d(q_i t_i)}{dq_i} = \epsilon \quad \ldots \ldots \ldots \quad (28)$$

for all the routes in use, where ϵ is a constant independent of the route number.

A route j will not be used if $\left[\dfrac{d(q_j t_j)}{dq_j} \right]_{q=0} \geqslant \epsilon$, for in that case, transfer to it of an increment δq from any route in use will increase the value of Z.

Now
$$\frac{d(q_i t_i)}{dq_i} = \frac{b_i}{\left(1 - \dfrac{q_i}{p_i}\right)^2} \quad \ldots \ldots \ldots \quad (29)$$

and so
$$\left[\frac{d(q_i t_i)}{dq_i} \right]_{q=0} = b_i \quad \ldots \ldots \ldots \quad (30)$$

Thus if $b_j < \epsilon < b_{j+1}$ only the first j routes will be in use.

Hence
$$1 - \frac{q_i}{p_i} = \sqrt{\frac{b_i}{\epsilon}}$$

and so
$$p_i - q_i = p_i \sqrt{\frac{b_i}{\epsilon}} \quad \text{where } i = 1, 2, \ldots j.$$

Summing over the first j routes gives :

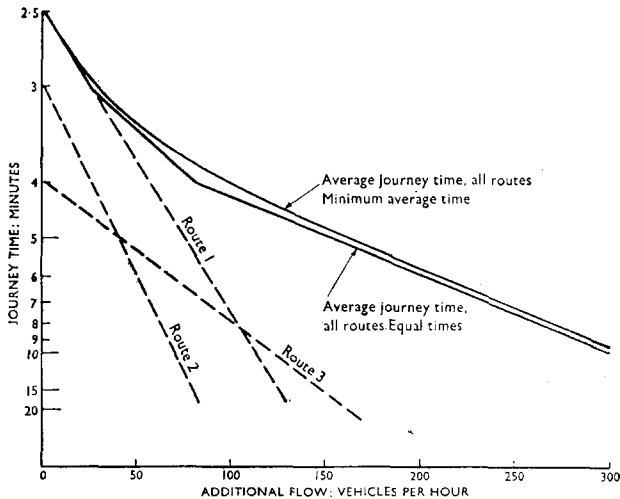$$\sum_{i=1}^{j} p_i - Q = \sum_{i=1}^{j} p_i \sqrt{\frac{b_i}{\epsilon}} \quad \ldots \ldots \quad (31)$$

With a few lines of algebra it can be shown that :

$$T = \frac{1}{Q} \left\{ \frac{\left(\sum\limits_{i=1}^{j} p_i \sqrt{b_i} \right)^2}{\sum\limits_{i=1}^{j} p_i - Q} - \sum\limits_{i=1}^{j} p_i b_i \right\} \quad \cdots \quad (32)$$

and the distribution of flows is given by :

$$q_h = p_h \left\{ 1 - \frac{\left(\sum\limits_{i=1}^{j} p_i - Q \right) \sqrt{b_h}}{\sum\limits_{i=1}^{j} p_i b_i} \right\}, \; h = 1, 2, \ldots j$$

$$q_h = 0, \qquad\qquad\qquad h = j + 1, \ldots D. \qquad (33)$$

Fig. 8

DISTRIBUTION OF TRAFFIC ON ALTERNATIVE ROUTES JOURNEY-TIME/FLOW
RELATIONS FOR A HYPOTHETICAL EXAMPLE WITH THREE ALTERNATIVES

The result of calculating the average journey time by each method for a particular hypothetical example is illustrated in *Fig.* 8. Here the average journey time is plotted, on an inverse scale, against the additional flow using some or all of the three routes. The inverse scale of journey time is used so that the individual relations between journey time and additional flow, also shown in the Figure, are linear. It will be seen that the advantage given by the minimum average method is not very great in this case.

Flows on a Network

In the case of a network of roads the theoretical problem becomes very complicated. However, it is possible to suggest a general line of approach. The same two criteria may be used as alternatives. That is to say, alternative routes between any two points may be chosen so that either all routes used have the same journey time, or else the average journey time for all journeys, and therefore the average number of vehicles on the network, is the minimum.

" BEFORE-AND-AFTER " STUDIES

Almost every traffic investigation involving a study of actual conditions is a " before-and-after " problem. Even general surveys and censuses are normally undertaken to discover trends in traffic conditions, which means making one comparison or a whole series of comparisons. Consider two sets of observations, one taken before and the other after a change. It is assumed that each " before " observation is chosen at random from a very large number of possible values (the population), and each choice is made independently of the others. The set of observations is then a random sample from the population. Similarly the " after " observations are assumed to form a random sample from a very large population, in general differing from the first population. The concepts of a random sample from a population and of significance, which is discussed below, are dealt with fully in most books on statistics, for example, Yule and Kendall.[10]

Suppose that the number of observations (the sample size) is n " before " and N " after," and the observations, which may be measurements of journey time, delay, speed, parking concentration or any other measurable quantity, are $x_1, x_2, \ldots x_n$ " before " and $X_1, X_2, \ldots X_N$ " after." The standard practice is to calculate the means :

$$x = \sum_{i=1}^{n} x_i/n \quad \text{and} \quad \overline{X} = \sum_{j=1}^{N} X_j/N$$

and to consider the difference between them, $d = \overline{X} - \bar{x}$, as an index of the change in the quantity measured. Now owing to purely random fluctuations, even if there is no difference between the populations, the means $\bar{x}$ and $\overline{X}$ may be expected to differ. In order to test whether the difference is likely to mean a " true " difference between the populations, it is necessary to consider the expected variation in x and $\overline{X}$. Suppose that the population means are m and M respectively, and that $\delta = M - m$. An estimate s of the standard deviation σ_x of the population of x's is calculated from the " before " sample ; similarly an estimate S of the standard deviation σ_X of the population of X's is found from the " after " sample. These two estimates are defined as follows :

$$s = \sqrt{\left\{ \frac{\sum\limits_{i=1}^{n} (x_i - \bar{x})^2}{n-1} \right\}}$$

$$S = \sqrt{\left\{ \frac{\sum\limits_{j=1}^{N} (X_j - \bar{X})^2}{N-1} \right\}} \right\} \quad \dots \dots \quad (34)$$

In order to test the significance of $d = \bar{X} - \bar{x}$, the estimated standard deviation of this quantity is calculated. This is :

$$s(d) = \sqrt{\left\{ \frac{s^2}{n} + \frac{S^2}{N} \right\}}$$

that is to say, $\quad s(d) = \sqrt{\left\{ \frac{\sum\limits_{i=1}^{n} (x_i - x)^2}{n(n-1)} + \frac{\sum\limits_{j=1}^{N} (X_j - \bar{X})^2}{N(N-1)} \right\}} \quad$. (35)

Now if the samples are of reasonable size and the populations have approximately " normal " distributions, which is often the case, the ratio $d/s(d)$ is also approximately a normal variable with unit standard deviation. If there is no difference between population means, that is to say, if $\delta = 0$, $d/s(d)$ also has an expected value of zero. Tables of a normal variable with zero mean and unit standard deviation are given in most books on statistics; a very complete Table is available.[11] The value found in the experiment can be looked up in such a Table and the probability that a difference at least as large (either positive or negative) would occur by chance can then be read off. If this probability (α) is less than 0·05 the difference d is said to be significantly different from zero (often abbreviated to " significant ") at the 5-per-cent level ; if it is less than 0·01, d is significant at the 1-per-cent level, and so on. The corresponding values of $d/s(d)$ are 1·96 and 2·58. The 5-per-cent level is the one most commonly used ; if used consistently in a long series of " before-and-after " studies it would mean that in all cases where there was no " true " difference the sample difference would be wrongly accounted as significant about once in 20 times.

Size of Sample

The size of sample in each case may be determined by the periods during which observations can be made, or, more likely, by the man-hours which can be spared. On other occasions the " before " study may be limited, with no possibility of extending the sample because an irrevocable change has been made, while the " after " sample can be much larger. It is useful here to consider the effect of increasing N, if n is fixed, on $s(d)$, the standard error of the difference, assuming that $\sigma_x = \sigma_X$. Expressed as a percentage of the value when $n = N$, the results are as follows :

Ratio of sample sizes ($N:n$)	Standard error of difference
1:1	100
2:1	87
4:1	79
10:1	74
100:1	71

Evidently it is not worth while to increase the " after " sample to more than 4 times the " before " sample, unless the observations are very easy to make.

Where the whole experiment is planned from the outset, and there is no limit to the sample sizes, the most efficient method, assuming that little difference between the population standard deviations is expected, is to have equal samples sizes, say n, before and after. It is clear from the form of the equation for the standard error of the difference that the greater the sample size the greater the precision of d. On the other hand the smallest sample means the least work. Some criterion is needed to decide the minimum sample size which is sufficient.

Suppose that the smallest difference of any concern is δ and that the significance level α is to be 5 per cent. If there is a real population difference equal to δ, there is still a possibility that the difference between a pair of samples will be rejected as non-significant. Suppose that the experimenter decides that he is willing to accept a 10-per-cent risk of making this mistake. Now in this case the probability that $-1.96 < d/s(d) < 1.96$, and therefore the sample difference is non-significant, is the same as the probability that :

$$-1.96 - \frac{\delta}{s(d)} < \frac{d-\delta}{s(d)} < 1.96 - \frac{\delta}{s(d)}$$

and this is to be 0·1. If δ is such that $1.96 - \dfrac{\delta}{s(d)} = -1.28$, that is to say, if $\dfrac{\delta}{s(d)} = 3.24$, the probability of the left-hand inequality is approximately 0·1, since $\dfrac{d-\delta}{s(d)}$ is approximately a normal variable * with zero mean and unit standard deviation. In this case the probability that $\dfrac{d-\delta}{s(d)} < -1.96 - \dfrac{\delta}{s(d)} = -5.20$ is negligible so that the condition required is

$$\frac{\delta}{s(d)} = 3.24$$

---

* Strictly $\dfrac{d-\delta}{s(d)}$ has a " t-distribution ", and the t-test should be used, but the error in assuming normality is small when sample sizes of 100 or more are involved and this is usually the case in traffic studies.

In terms of the sample size and the standard error of a single observation, σ, assumed to be the same before and after, this gives

$$\frac{\delta}{\sigma\sqrt{\frac{2}{n}}} = 3 \cdot 24$$

or

$$n = 21 \cdot 0 \left(\frac{\sigma}{\delta}\right)^2$$

In general, the sample size is

$$n = \gamma \left(\frac{\sigma}{\delta}\right)^2 \quad \dotso \quad (36)$$

where γ depends on the significance level α and the acceptable probability β of rejecting a true difference δ. Values of γ for some typical values of

TABLE 2.—VALUES OF γ IN FORMULA $n = \gamma \left(\dfrac{\sigma}{\delta}\right)^2$ FOR ASSESSING SAMPLE

SIZE

Probability of rejecting "true" difference (β) : per cent	Significance level, α : per cent		
	5	2	1
10	21·0	26·0	29·8
5	26·0	31·5	35·6
2	32·2	38·4	42·9
1	36·7	43·3	48·1

α and β are given in Table 2. In order to use this Table some estimate of the standard error of a single variable is required. Generally previous observations provide this; otherwise a guess must be made, and the sample size corrected, if necessary, in the light of the first results.

As an example, consider a " before-and-after " study of journey times. The ratio of the standard deviation to the mean is often about 1:4, so that if the significance level is 5 per cent, and the probability of rejecting a 10-per-cent difference is also to be 5 per cent, the sample size is

$$n = 26 \cdot 0 \left(\frac{\sigma}{\delta}\right)^2 = 26 \cdot 0 (2 \cdot 5)^2 = 160 \text{ approximately.}$$

That is, samples of about 160 runs before and after are required.

Several Routes and Several Periods

It is often necessary to consider the effect of a change in road lay-out or method of control on journey time on a number of routes. For instance, the whole effect of a one-way system can only be found by studying every route affected by the system. Similarly it may be necessary to cover

different periods in both the "before" and the "after" experiment, because the effect which is sought varies with time of day, day of week, or season of year. Methods of combining results for several routes and several periods are discussed in Appendix VI.

Average Journey Time, Distance, and Speed

In summarizing a "before-and-after" study it is often necessary to give comparative figures for journey time, distance, and speed, averaged over all periods and times. This is particularly desirable in the case of one-way systems. Here the average speed on a journey usually increases, but the average distance travelled per journey also increases, owing to the less direct routes followed. If the average speed increases by a greater proportion than the average distance, the average journey time is reduced, and there is a saving in vehicle-hours. On the other hand the increase in speed may be proportionally less than that of distance, so that the average journey time is longer and vehicle-hours are wasted. The appropriate average values to reveal this effect are given below.

Suppose that the various quantities measured for a typical route during a typical period are expressed symbolically as follows :

Duration of period		Length of route	Flow	Mean journey time
θ	Before	l	q	t
	After	L	Q	T

Then it is shown in Appendix VI that the overall weighted average journey time " before " is :

$$\bar{t}_w = \Sigma(q + Q)\theta t / \Sigma(q + Q)\theta \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where the summations extend over all routes and periods.

The average distance per journey " before," weighted in the same way is :

$$\bar{l}_w = \Sigma(q + Q)\theta l / \Sigma(q + Q)\theta \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The average space-mean speed " before " is :

$$\bar{v}_s = \bar{l}_w / \bar{t}_w = \Sigma(q + Q)\theta l / \Sigma(q + Q)\theta t \quad . \quad . \quad . \quad (39)$$

The corresponding quantities for the " after " results are obtained by replacing lower case by capital letters in these expressions, the term $q + Q$ remaining unaltered.

Two examples of results for one-way systems treated in this way are shown in Table 3.

It will be seen that in the first case the increase in distance outweighs the increase in speed, and the average journey time is increased ; in the second case, however, the speed change dominates and the journey time is reduced.

TABLE 3.—AVERAGE EFFECTS OF TWO ONE-WAY SYSTEMS

Site of one-way system		Average distance per journey: miles	Average journey speed: m.p.h.	Average journey time: minutes
Slough. . .	Before	0·43	13·8	1·88
	After	0·52	15·2	2·06
	Change: per cent	+ 21	+ 10	+ 10
Holland Road (London)	Before	0·42	13·2	1·93
	After	0·46	15·9	1·73
	Change: per cent	+ 10	+ 20	− 10

Vehicle-Mileage and Vehicle-Hours

It should be noted that the quantities which appear as denominators and numerators in the expressions for average journey time, distance, and speed all have a direct physical meaning. It is pointed out in Appendix VI that, for the typical route and period, $\frac{1}{2}(q + Q)$ is taken as an estimate of the average flow, assumed to be the same before and after. It follows that, for the " before " half of the experiment, for instance :

$\frac{1}{2}\Sigma(q + Q)\theta$ = total vehicles using routes during periods studied ;

$\frac{1}{2}\Sigma(q + Q)\theta l$ = total vehicle-mileage on routes during periods studied ;

and $\frac{1}{2}\Sigma(q + Q)\theta t$ = total vehicle-hours on routes during periods studied.

Hence :   mean length of journey $= \dfrac{\text{vehicle-miles}}{\text{vehicles}}$

            mean journey time $= \dfrac{\text{vehicle-hours}}{\text{vehicles}}$

and   space-mean speed $= \dfrac{\text{vehicle-miles}}{\text{vehicle-hours}}.$

As was pointed out earlier, if very sensitive statistical tests are required it may be better to apply them to the averages of the speeds of individual vehicles rather than to average journey times. However, the resulting quantities have no direct physical meaning, and should not be used to predict changes in vehicle-hours. The correct course is to use speeds, if necessary, for statistical tests, but journey times in all cases for assessing gains or losses.

FUTURE WORK

This Paper has presented a number of theoretical considerations and formulae which are relevant to the practical problems arising in traffic engineering research. A great deal more work is needed, both in the theoretical field and in the application of these and similar methods in actual cases. Many of the results given can easily be extended to more complicated situations, but this is not always possible. More work is required to find the theoretical relation between delay and flow at intersections, particularly those with vehicle-actuated signals, and the effect of linking signal installations, and to verify any results obtained. Another question of importance is that of the effect of the high correlation between the journey times of two vehicles following a given route within a very short interval; this means, for example, that a sample of journey times of successive vehicles has not the same value as a sample of equal size taken at intervals of, say, 10 minutes.

CONCLUSION

It has been demonstrated that many traffic problems involve theoretical considerations and that a knowledge of elementary statistics is desirable for work on traffic research. It is hoped that the results and ideas given in this Paper will be helpful to road engineers when dealing with the types of investigation discussed, and will stimulate further research on the subject.

ACKNOWLEDGEMENT

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The Paper is accompanied by eight sheets of drawings, from which the Figures in the text have been prepared, and by six Appendices.

APPENDIX I

GLOSSARY AND SYMBOLS

Headway.—The distance from the front of a vehicle to the front of the one directly ahead.

Flow.—The number of vehicles passing a given point on the road per unit time, in one direction or both according to the context. (q).

Saturation flow (at traffic signals).—The flow which is reached when vehicles are travelling at minimum achievable headway. (p).

Speed.—(v).

Journey speed.—The average speed on a journey, including stops.

Running-speed.—The average speed while in motion. (v_r).

Concentration.—The number of vehicles per unit length of road. (k).

Time-distribution of speed.—The frequency distribution of speeds of vehicles in time as they pass a point on the road. The subscript t is used for the time-distribution.

Space-distribution of speed.—The frequency distribution of speeds of vehicles on a uniform road at an instant of time. The subscript s is used for the space-distribution.

Time-mean speed.—The mean of the time-distribution of speed. ($\bar{v_t}$).

Space-mean speed.—The mean of the space-distribution of speed. ($\bar{v_s}$).

" Green " time (at traffic signals).—The time during which the signal for a particular phase is green or amber. (g).

" Red " time (at traffic signals).—The time during which the signal for a particular phase is red or red/amber. (r).

" Lost " time (at traffic signals).—The part of the " green " time for a particular phase which is effectively " lost ", chiefly due to acceleration delay. (a). It is approximately true that no vehicles leave the intersection at the beginning of the " green " time until the lost time has elapsed, after which they leave at the saturation flow.

Effective red time (at traffic signals).—The " red " time plus the " lost " time.

Delay (at traffic signals).—The difference between the average journey time through the intersection and the time for a run which is not stopped or slowed down by the signals. (Average delay : t).

Journey time.—The time on a journey including stops.

Note. The symbols shown above may appear in the text in a modified form, for example, with subscripts or as capitals, with meanings which are defined in the context.

APPENDIX II

RELATION BETWEEN TIME- AND SPACE-MEAN SPEEDS

The time-mean speed is defined by equation (2) as follows:

$$\bar{v}_t = \sum_{i=1}^{C} q_i v_i / Q$$

where q_i denotes the flow of the subsidiary stream whose speed is v_i and Q denotes the total flow. The space-mean speed is given by equation (3) as:

$$\bar{v}_s = \sum_{i=1}^{C} k_i v_i / K$$

where k_i denotes the concentration of the subsidiary stream whose speed is v_i and K denotes the total concentration. Substituting for the value of q_i (as given in equation (4)) in the expression for $\bar{v}_t$ gives

$$\bar{v}_t = \sum_{i=1}^{C} k_i v_i^2 / Q$$

$$= K \sum_{i=1}^{C} f_i' v_i^2 / Q \quad \text{by the definition of } f_i'$$

$$= \sum_{i=1}^{C} f_i' v_i^2 / \bar{v}_s \quad \text{from (5)}$$

$$= \sum_{i=1}^{C} f_i' \{\bar{v}_s + (v_i - \bar{v}_s)\}^2 / \bar{v}_s$$

$$= \left\{ \left(\sum_{i=1}^{C} f_i' \right) \bar{v}_s^2 + \sum_{i=1}^{C} f_i' (v_i - \bar{v}_s)^2 \right\} / \bar{v}_s$$

$$\left[\text{since } \sum_{i=1}^{C} f_i' (v_i - \bar{v}_s) = 0 \right]$$

$$= \bar{v}_s + \frac{\sigma_s^2}{\bar{v}_s},$$

where $\sigma_s = \sqrt{\left\{ \sum_{i=1}^{C} f_i' (v_i - \bar{v}_s)^2 \right\}}$ is the standard deviation of the space distribution of speed. This relation shows that $\bar{v}_t \geqslant \bar{v}_s$ on all occasions, and if there is any variation in speed at all $\sigma_s > 0$ and $\bar{v}_t > \bar{v}_s$.

APPENDIX III

MINIMUM CYCLE AND MEAN DELAY FOR A FIXED-TIME TRAFFIC SIGNAL, ASSUMING UNIFORM FLOW

The formulae given below have been derived in substantially the same form by other authors. In particular the method of derivation follows that of Clayton [7], but it is given here for the sake of completeness.

The symbols used are defined on p. 15. The cycle is minimum when each green period is saturated, that is to say, when vehicles leave at minimum headway up to the end of the period but no vehicles are held over to the next cycle. In phase 1, the period a_1 is lost, so that the number of vehicles cleared is $p_1(g_1 - a_1)$. But this must equal the number arriving during one cycle, q_1c. Hence:

$$p_1(g_1 - a_1) = q_1c,$$

that is to say,

$$g_1 = a_1 + \frac{q_1}{p_1}c.$$

Similarly,

$$g_2 = a_2 + \frac{q_2}{p_2}c.$$

Adding these two last equations gives:

$$c = A + \left(\frac{q_1}{p_1} + \frac{q_2}{p_2}\right)c,$$

that is to say,

$$c = \frac{A}{1 - \frac{q_1}{p_1} - \frac{q_2}{p_2}} \quad \text{(Adams' Formula)}.$$

In the general case, if the cycle exceeds this minimum the condition must be satisfied that the number of vehicles which arrive during one cycle is less than the maximum number which could be cleared (that is to say, if the green period was saturated for each phase). That is:

$$q_1c \leqslant p_1(g_1 - a_1) \text{ and } q_2c \leqslant p_2(g_2 - a_2)$$

These conditions provide upper limits to the two flows.

Consider the delay to successive vehicles, remembering that the flow is assumed to be uniform. *Fig. 4* shows that successive vehicles leave the intersections at times a, $a + \frac{1}{p}$, $a + \frac{2}{p}$, ... after the appearance of the green, until the period of saturation flow ceases. Since vehicles arrive at intervals $1/q$, the first vehicle to be stopped would have cleared the intersection at any time between 0 and $1/q$ after the beginning of the red period if it had not been stopped. Suppose that it actually arrives at such a time that it would have cleared the intersection at the average time $1/2q$ after the beginning of the red, as shown in *Fig. 4*. Then it is delayed by an amount $r + a - \frac{1}{2q}$. The next vehicle arrives an interval $1/q$ later, and leaves $1/p$ later, than the first, so that its delay is $\frac{1}{q} - \frac{1}{p}$ less. Similarly each successive vehicle is delayed by $\frac{1}{q} - \frac{1}{p}$ less than the previous one. Suppose that the average delay to the last delayed vehicle is $\frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)$; this is reasonable, since it is equally likely to be anywhere between 0 and $\frac{1}{q} - \frac{1}{p}$. Suppose also that the number of vehicles delayed is n. The average delay to delayed vehicles is the average of the delays to the first and the last. Evidently the first vehicle is delayed by an amount $\frac{2n-1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)$ so that the average delay to delayed vehicles is $\frac{1}{2}n\left(\frac{1}{q} - \frac{1}{p}\right)$. But it has already been shown that the average delay to the first vehicle can be expressed as $r + a - \frac{1}{2q}$, and equating the two expressions for this quantity gives:

$$n = \left(r + a - \frac{1}{2p}\right)\Big/\left(\frac{1}{q} - \frac{1}{p}\right)$$

The total delay is $\frac{1}{2}n^2\left(\frac{1}{q} - \frac{1}{p}\right)$ and the average for all vehicles is:

$$t = n^2\left(\frac{1}{q} - \frac{1}{p}\right)\Big/ qc$$

that is to say,
$$t = \frac{\left(r + a - \dfrac{1}{2p}\right)^2}{2c\left(1 - \dfrac{q}{p}\right)}$$

Formulae differing slightly from this can be obtained by altering the assumptions, but when evaluated the differences are generally trivial.

APPENDIX IV

Optimum Phase Times for a Fixed-Time Two-Phase Traffic Signal

The average delay for the intersection as a whole is given by equation (16), which reads :

$$T = \frac{1}{2Qc}\left\{\frac{q_1(r_1 + a_1)^2}{1 - \dfrac{q_1}{p_1}} + \frac{q_2(r_2 + a_2)^2}{1 - \dfrac{q_2}{p_2}}\right\}$$

Let
$$r_1 + a_1 = xc$$
and
$$r_2 + a_2 = yc.$$

Addition gives
$$c + a = (x + y)c,$$

therefore
$$c = \frac{a}{x + y - 1}.$$

In terms of x and y,
$$T = \frac{A}{2Q(x + y - 1)}\left\{\frac{q_1 x^2}{1 - \dfrac{q_1}{p_1}} + \frac{q_2 y^2}{1 - \dfrac{q_2}{p_2}}\right\}$$

The conditions given on p. 357 which ensure that vehicles do not accumulate indefinitely on either phase, become :

$$x \leqslant \xi < 1$$
and
$$y \leqslant \eta < 1,$$

where
$$\xi = 1 - \frac{q_1}{p_1}$$

and
$$\eta = 1 - \frac{q_1}{p_1}$$

Let
$$\lambda = Aq_1/2Q\xi$$
and
$$\mu = Aq_2/2Q\eta$$

then
$$T = (\lambda x^2 + \mu y^2)/(x + y - 1)$$

Now it is generally assumed that the optimum cycle is the shortest one. The shortest cycle occurs when x and y have their maximum values, that is to say, when $x = \xi$ and $y = \eta$. In order to test whether t is a minimum at this point, consider the values of $\dfrac{\partial T}{\partial x}$ and $\dfrac{\partial T}{\partial y}$ when $x = \xi$ and $y = \eta$.

$$\left[\frac{\partial T}{\partial x}\right]_{\xi, \eta} = \frac{\lambda\xi^2 - \mu\eta^2 - 2\lambda\xi(1 - \eta)}{(\xi + \eta - 1)^2}$$

and
$$\left[\frac{\partial T}{\partial y}\right]_{\xi, \eta} = \frac{\mu\eta^2 - \lambda\xi^2 - 2\mu\eta(1 - \xi)}{(\xi + \eta - 1)^2}$$

Now the only possible changes in x and y in this case are reductions. Hence T is minimum when $x = \xi$, $y = \eta$, if $\quad\dfrac{\partial T}{\partial x} < 0, \dfrac{\partial T}{\partial y} < 0$

Now
$$(\xi + \eta - 1)^2 \left[\frac{\partial T}{\partial y}\right]_{\xi,\eta} = \mu(\eta^2 - 2\eta + 2\xi\eta) - \lambda\xi^2$$
$$= -\mu\{2\eta(1 - \eta) + (\xi + \eta)^2\} - (\lambda - \mu)\xi^2$$

Suppose that $\lambda \geqslant \mu$. If this condition is not satisfied, phases 1 and 2 can be interchanged, and it will then be satisfied. Then:
$$\left[\frac{\partial T}{\partial y}\right]_{\xi,\eta} < 0$$
since the right-hand side of the above expression is negative while $(\xi + \eta - 1)^2$ is positive. In this case, therefore, the condition that T is the minimum is:
$$\left[\frac{\partial T}{\partial x}\right]_{\xi,\eta} < 0$$

that is to say, $\lambda\xi^2 - \mu\eta^2 - 2\lambda\xi(1 - \eta) < 0$

or $(\mu/\lambda)\eta^2 + 2\xi(1 - \eta) - \xi^2 > 0$

If, however, $(\mu/\lambda)\eta^2 + 2\xi(1 - \eta) - \xi^2 < 0$, T is a minimum at (x, η), where x is given by $\left[\dfrac{\partial T}{\partial x}\right]_{\xi,\eta} = 0$,

that is to say, $\lambda x^2 - \mu\eta^2 - 2\lambda x(1 - \eta) = 0$

The root of this equation, in the range $0 < x < \eta$, is:
$$1 - \eta + \sqrt{\{(1 - \eta)^2 + (\mu/\lambda)\eta^2\}}$$

In terms of the original quantities, the condition $\lambda \geqslant \mu$ becomes :
$$\left(\frac{1}{q_1} - \frac{1}{p_1}\right) \geqslant \left(\frac{1}{q_2} - \frac{1}{p_2}\right)$$

This determines which phase shall be called 1 and which 2. The condition that the minimum cycle shall be the optimum is
$$q_1\left(1 - \frac{q_1}{p_1}\right) - q_2\left(1 - \frac{q_2}{p_2}\right) < \frac{2q_1q_2}{p_2}.$$

If this condition is not satisfied, the minimum value of T occurs when
$$c = \frac{A}{\sqrt{\{(1 - \eta)^2 + (\mu/\lambda)\eta^2\}}}$$
$$= \frac{A}{\sqrt{\left\{\left(\dfrac{q_2}{p_2}\right)^2 + \dfrac{q_2}{q_1}\left(1 - \dfrac{q_1}{p_1}\right)\left(1 - \dfrac{q_2}{p_2}\right)\right\}}}$$

APPENDIX V

DELAY FOR AN "IDEAL" VEHICLE-ACTUATED TRAFFIC SIGNAL

Here the cycle is assumed to be the minimum, so that :
$$c = \frac{A}{1 - \dfrac{q_1}{p_1} - \dfrac{q_2}{p_2}}$$
$$r_1 + a_1 = \left(1 - \frac{q_1}{p_1}\right)c$$
and $r_2 + a^2 = \left(1 - \dfrac{q_2}{p_2}\right)c$

Substituting these values in equations (14) and (16) gives the following equations for average delays t_1 and t_2 on each phase and T for the intersection as a whole:

$$t_1 = \tfrac{1}{2}c\left(1 - \frac{q_1}{p_1}\right) = \tfrac{1}{2}A\left(1 - \frac{q_1}{p_1}\right)\Big/\left(1 - \frac{q_1}{p_1} - \frac{q_2}{p_2}\right)$$

$$t_2 = \tfrac{1}{2}c\left(1 - \frac{q_2}{p_2}\right) = \tfrac{1}{2}A\left(1 - \frac{q_2}{p_2}\right)\Big/\left(1 - \frac{q_1}{p_1} - \frac{q_2}{p_2}\right)$$

and $$T = \frac{A}{2Q\left(1 - \frac{q_1}{p_1} - \frac{q_2}{p_2}\right)}\left\{q_1\left(1 - \frac{q_1}{p_1}\right) + q_2\left(1 - \frac{q_2}{p_2}\right)\right\}$$

In the particular case where the intersection is fully symmetrical, so that $q_1 = q_2 = q$, $p_1 = p_2 = p$, and $a_1 = a_2 = a$, the average delay on each phase is:

$$t = \frac{a\left(1 - \frac{q}{p}\right)}{1 - \frac{2q}{p}}$$

APPENDIX VI

COMPARISON OF JOURNEY TIMES IN " BEFORE-AND-AFTER " STUDIES WITH SEVERAL ROUTES AND WITH SEVERAL PERIODS

Several Routes

Suppose there are E routes, and the results of the experiment are symbolically as follows :

Route		Flow	Sample size	Mean journey time	Standard deviation of journey times
i	Before	q_i	n_i	t_i	s_i
	After	Q_i	N_i	T_i	S_i

It is desired to combine the " before " results and compare them with the " after " results. A simple method is to find the arithmetic mean of the mean journey times on the routes, that is to say :

$$\bar{t} = \sum_{i=1}^{E} t_i/E, \quad \bar{T} = \sum_{i=1}^{E} T_i/E$$

The standard error of $\bar{T} - \bar{t}$ is then :

$$s(\bar{T} - \bar{t}) = \frac{1}{E}\sqrt{\left\{\sum_{i=1}^{E}\left(\frac{s_i^2}{n_i} + \frac{S_i^2}{N_i}\right)\right\}}$$

and the usual test for significance can be applied.

However, if the flows on the various routes are very different, it is better to attach weights to the journey times to allow for this. If the experiment has been properly planned, the difference between the measured flows on any route i before and after will not be significant. Otherwise it must be assumed that there is a real difference in the flow, and this would itself be expected to affect journey times and spoil the experiment. The two flows can therefore be pooled to give an average flow of $\tfrac{1}{2}(q_i + Q_i)$. These average flows can then be used as weights, giving weighted average journey times :

$$\bar{t}_w = \sum_{i=1}^{E} (q_i + Q_i)t_i \bigg/ \sum_{i=1}^{E} (q_i + Q_i)$$

$$\bar{T}_w = \sum_{i=1}^{E} (q_i + Q_i)T_i \bigg/ \sum_{i=1}^{E} (q_i + Q_i)$$

The factor $\frac{1}{2}$ cancels out between denominator and numerator. It is essential for a fair comparison that the weights used should be the same before and after. The standard error of $\bar{T}_w - \bar{t}_w$ is :

$$s(\bar{T}_w - \bar{t}_w) = \frac{1}{\displaystyle\sum_{i=1}^{E} (q_i + Q_i)} \sqrt{\left\{ \sum_{i=1}^{E} (q_i + Q_i)^2 \left(\frac{s_i^2}{n_i} + \frac{S_i^2}{N_i} \right) \right\}}$$

and this is used for testing the significance of $\bar{T}_w - \bar{t}_w$.

Several Periods

Another common case occurs when journey times on a single route are studied at different times of the day and/or on different days of the week. Suppose that there are F periods, each period being distinguished from the others because of differences in conditions. It is essential to cover the same periods (that is to say, the same hours of the same days of the week) " before " and " after." Suppose that in each period a sample number of runs is taken, which can be regarded as representative of the period as a whole. Let the results be symbolically as follows :

Period	Duration of period	Flow	Sample size	Mean journey time	Standard deviation of journey times
j	$\theta_j \begin{cases} \text{Before} \\ \text{After} \end{cases}$	q_j Q_j	n_j N_j	t_j T_j	s_j S_j

As before, the arithmetic mean of the journey times may be used, but again it is more reasonable to introduce weights. Here, since both the flows and the periods vary, it is appropriate to weight by the number of vehicles concerned in each period. Using the average flow as before, this is $\frac{1}{2}(q_j + Q_j)\theta_j$ for period j. The corresponding weighted mean journey times are :

$$\bar{t}_w = \sum_{j=1}^{F} (q_j + Q_j)\theta_j t_j \bigg/ \sum_{j=1}^{F} (q_j + Q_j)\theta_j$$

$$\text{and} \qquad \bar{T}_w = \sum_{j=1}^{F} (q_j + Q_j)\theta_j T_j \bigg/ \sum_{j=1}^{F} (q_j + Q_j)\theta_j$$

and the standard error of the difference is :

$$s(T_w - t_w) = \frac{1}{\displaystyle\sum_{j=1}^{F} (q_j + Q_j)\theta_j} \sqrt{\left\{ \sum_{j=1}^{F} (q_j + Q_j)^2 \theta_j^2 \left(\frac{s_j^2}{n_j} + \frac{S_j^2}{N_j} \right) \right\}}$$

Several Periods and Several Routes

If there are E routes, $i = 1, 2, \ldots E$, and F periods, $j = 1, 2, \ldots F$, then with an obvious notation the overall weighted mean journey time " before " is :

$$t_w = \sum_{i=1}^{E} \sum_{j=1}^{F} (q_{ij} + Q_{ij})\theta_j t_{ij} \bigg/ \sum_{i=1}^{E} \sum_{j=1}^{F} (q_{ij} + Q_{ij})\theta_j$$

$\overline{T}_w$ has a similar expression and the standard error of the difference is:

$$s(\overline{T}_w - \overline{t_w}) = \dfrac{1}{\displaystyle\sum_{i=1}^{E} \sum_{j=1}^{F} (q_{ij} + Q_{ij})\theta_j} \Bigg/ \sqrt{\left\{ \sum_{i=1}^{E} \sum_{j=1}^{F} (q_{ij} + Q_{ij})^2\theta_j{}^2 \left(\dfrac{s_{ij}{}^2}{n_{ij}} + \dfrac{S_{ij}{}^2}{N_{ij}} \right) \right\}}$$

The expression for $\overline{t}_w$ can be written in a simplified form as follows :

$$\overline{t}_w = \Sigma(q + Q)\theta t / \Sigma(q + Q)\theta$$

where the symbols on the right-hand side now refer to a typical route during a typical period, and the summation is understood to extend over all routes and periods. The expression for $\overline{T}_w$ can be treated in a similar way.

Discussion

The Author introduced the Paper with the aid of a series of lantern slides.

Dr W. H. Glanville said that he proposed to make only a few general remarks. First, he wanted to emphasize the importance of the kind of investigation into the theoretical side of road traffic which the Author had made. Some engineers were frightened when mathematics went beyond a fifth-form standard, but he could assure anyone who was willing to take the trouble that it was worth while to study the Paper in detail.

The Author had said that it was not always appreciated that in a severely practical subject such as traffic engineering, there was need for theory, and " the history of science suggests that progress in any field of research can best be achieved by a judicious mixture of practical experience, experiment, and theory." Dr Glanville considered that the Author had put that very well indeed. However, in introducing his Paper as a theoretical Paper he had been in a sense wrong, because, as Dr Glanville saw it, it was essentially a practical Paper making use of theory. What the Author had done was to advance from the pioneer investigations which had been made some years ago by Mr Adams, Mr Clayton, and others.

There was a very wide scope for investigations, theoretical and otherwise, into traffic movement, and theory could help research considerably, and therefore could help practice in many ways, for example in planning and analysing experiments, in generalizing from the experimental results, in solving problems which it would be too costly to treat experimentally, and not the least in providing an outlook which is critical of current ideas and receptive to new ones, but, of course, equally critical of them.

For instance, in connexion with the investigations conducted by the Road Research Laboratory, it was of great help to be able to investigate those matters theoretically, and it helped them to cut down the man-power required in a research. For example, from what was said on pp. 348–351 of the Paper it would be seen that a large number of runs by a test car had